Available online at www.sciencedirect.com

SCIENCE DIRECT

Journal of Approximation Theory 132 (2005) 224–240

Journal of Approximation Theory

www.elsevier.com/locate/jat

On the distribution of poles of Padé approximants to the Z-transform of complex Gaussian white noise

P. Barone*

Instituto per le Applicazioni del Calcolo "M. Picone", C.N.R., Viale del Policlinico 137, 00161 Rome, Italy

Received 11 February 2004; received in revised form 25 September 2004; accepted in revised form 27 October 2004

Communicated by Guillermo López Lagomasino
Available online 7 January 2005

Abstract

In the application of Padé methods to signal processing a basic problem is to take into account the effect of measurement noise on the computed approximants. Qualitative deterministic noise models have been proposed which are consistent with experimental results. In this paper the Padé approximants to the Z-transform of a complex Gaussian discrete white noise process are considered. Properties of the condensed density of the Padé poles such as circular symmetry, asymptotic concentration on the unit circle and independence on the noise variance are proved. An analytic model of the condensed density of the Padé poles for all orders of the approximants is also computed. Some Monte Carlo simulations are provided.

Keywords: Padé approximants; Logarithmic potentials; Random determinants; Random polynomials

0. Introduction

In many signal processing problems a finite number of terms of the sequence of equispaced data

$$a_k = s_k + v_k, \quad k = 0, 1, 2, \ldots \tag{1}$$

* Fax: +39 06 4404306.
E-mail address: barone@iac.rm.cnr.it (P. Barone).

0021-9045/$ - see front matter
doi:10.1016/j.jat.2004.10.014

where v_k is a complex Gaussian white noise sequence is given. The Z-transform

$$s(z) = \sum_{k=0}^{\infty} s_k z^{-k}$$

is then considered and inference on $\{s_k\}$ is performed based on a Padé approximant $[l, n]_f$ of $f(z) = \sum_{k=0}^{\infty} a_k z^{-k}$ computable from the observations $a_k,\ k = 0, \ldots, l+n$. The poles of $[l, n]_s$ in the complex plane are the key quantities to make inference on $\{s_k\}$. Unfortunately these quantities are very sensitive to noise. Therefore the knowledge of an approximation of $f(z)$ alone hardly helps to solve the problem. However, the appearance of the Froissart doublets [10] (pole-zero pairs of $[l, n]_f$ close to the unit circle at a distance proportional to the scale of the noise) can help identifying the poles of $[l, n]_s$ filtering out the noise-related ones [6]. Following a somewhat different approach in [2–4,15,16], we were able to get some partially rigorous asymptotic results on the behavior of the poles of $[l, n]_f$ for some specific classes of functions $s(z)$ by using qualitative models for the Z-transform $v(z)$ of the noise. Moreover, we demonstrated experimentally that inference methods based on these asymptotic results perform very well in many real life problems. The aim of this paper is to provide some theoretical justification for using such qualitative noise models.

More specifically the qualitative models for $v(z)$ are derived as follows. First notice that if the v_k are independent and identically distributed complex random variables with mean zero and variance σ^2, the variance of $v(z)$ is [1]

$$E\{|v(z)|^2\} = \sum_{k=0}^{\infty} E\{|v_k z^{-k}|^2\} = \sigma^2 \sum_{k=0}^{\infty} |z|^{-2k} = \sigma^2/(1-|z|^{-2}),$$

where $E\{\cdot\}$ denotes the expected value and the series converges in quadratic mean for $|z| > 1$.

The variance diverges as z approaches the unit circle in the complex plane. Thus the presence of the noise makes the unit circle a natural boundary for the function $f(z)$, i.e. a curve of singularities which separates the complex plane into two disconnected pieces. Then Gammel [8,9], motivated by the Froissart experimental results, conjectured that the noise function $v(z)$ is a quasi-analytic function in the sense of Carleman [7]. Following this conjecture, in the papers quoted before we then approximated the random function $v(z)$ by the following deterministic one:

$$\tilde{v}(z) = \sum_{r=1}^{m_v} B_r/(1-\beta_r z^{-1}), \tag{2}$$

where m_v is a large integer, the poles $\beta_r = e^{i\vartheta_r}$ are equally spaced along the unit circle, and the B_r are complex constants which satisfy the condition

$$|B_r| < \text{ const. } \exp(-r^{1+\gamma}),$$

with $\gamma > 0$. As $m_v \to \infty$ Carleman has shown [7] that function (2) converges to a quasi-analytic function which has the unit circle as a natural boundary. We would like to show that

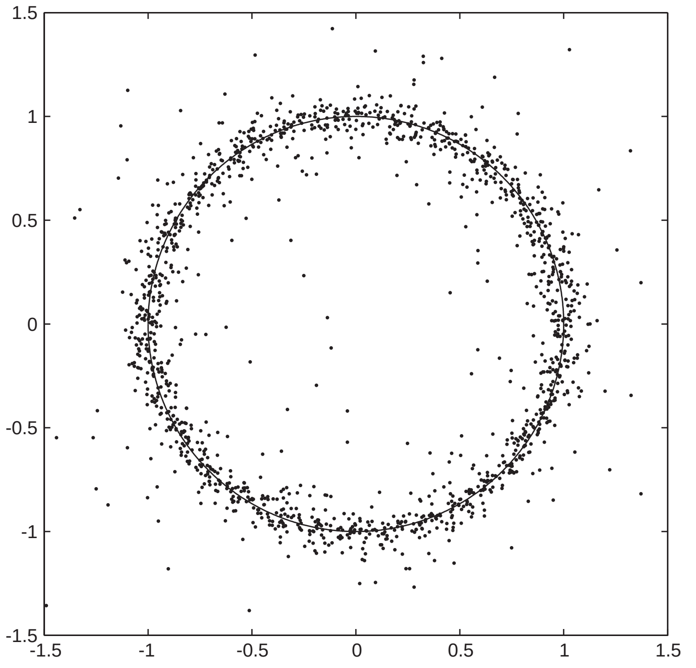

Fig. 1. Location of the poles of $[29, 30]_v(z)$ for 50 independent realizations of v.

the Padé approximants of $v(z)$ are random rational functions whose poles have a distribution concentrated close to the unit circle (see Fig. 1). In [18] the asymptotic zero distribution of polynomials orthogonal with respect to certain positive measures on the unit circle is studied and it is proved that the asymptotic zeros distribution is uniform on the unit circle. Unfortunately, this kind of result is not useful in the present context because the $f(z)$ is a random function, therefore the Padé approximants $[l, n]_f$ involve random polynomials orthogonal in a generalized sense with respect to the random data $\{a_k\}$ and hence the a_k are not the Fourier–Stieltjes coefficients of a positive measure.

In the following, it will be proved that the condensed density of the poles of $[l, n]_f$—i.e. the expectation of the normalized counting measure defined on the poles—when $s_k = 0$ and v_k are i.i.d. standard complex Gaussian variates is circularly symmetric for all n, it is concentrated around the unit circle, therefore supporting the choice of the deterministic model (2) and it is independent of the noise variance. Moreover an analytic model of the marginal condensed density, with respect to the absolute value of the poles, is derived for all orders n.

The paper is organized as follows. In Section 1 the properties of the condensed density of the Padé poles are derived. In Section 2 a model of the condensed density is computed. In Section 3 some numerical results are shown.

1. Properties of the condensed density

Let $f(z)$ be the Z-transform of the sequence $\{a_k\}$ given by the Laurent power series

$$f(z) = \sum_{k=0}^{\infty} a_k z^{-k}. \tag{3}$$

From the coefficients in this expansion we may compute the Padé approximant $[l,n]_f$ of orders l and n to $f(z)$ which is defined by [1]

$$[l,n]_f = P_1(z^{-1})/P_2(z^{-1}),$$

where $P_1(z)$ and $P_2(z)$ are polynomials in z of degree l and n, respectively, which satisfy

$$P_1(z^{-1}) - P_2(z^{-1})f(z) = O(z^{-(l+n+1)}), \quad \text{as } |z| \to \infty. \tag{4}$$

Given the a_k, $k = 0,\dots,l+n$, the substitution of expansion (3) in condition (4) yields a linear system of equations which determines the coefficients of the polynomials P_1 and P_2. If we write the coefficients of $P_1(z)$ and $P_2(z)$ as

$$P_1(z) = \beta_0 + \beta_1 z + \cdots + \beta_l z^l,$$

$$P_2(z) = 1 + \alpha_n z + \cdots + \alpha_1 z^n,$$

the linear system is given by [1]

$$\begin{aligned}
\beta_0 &= a_0,\\
\beta_1 &= a_1 + a_0\alpha_n,\\
\beta_2 &= a_2 + a_1\alpha_n + a_0\alpha_{n-1}\\
\vdots\ &\ \ \vdots\\
\beta_l &= a_l + a_{l-1}\alpha_n + \cdots + a_0\alpha_{n-l+1},\\
0 &= a_{l+1} + a_l\alpha_n + \cdots + a_{l-n+1}\alpha_1,\\
\vdots\ &\ \ \vdots\\
0 &= a_{l+n} + a_{l+n-1}\alpha_n + \cdots + a_l\alpha_1
\end{aligned}$$

with the convention that $a_j = 0$ if $j < 0$. Let us define for every integer q the Hankel matrices

$$U^{(q)} = \begin{pmatrix} a_q & a_{q+1} & \dots & a_{q+n-1} \\ a_{q+1} & a_{q+2} & \dots & a_{q+n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{q+n-1} & a_{q+n} & \dots & a_{q+2n-2} \end{pmatrix} \tag{5}$$

and the vectors $\underline{a}^{(q)} = [a_{q+n},\dots,a_{q+2n-1}]^T$. To compute the coefficients $\underline{\alpha}$ of $P_2(z)$ we have then to solve the linear system

$$U^{(l-n+1)}\underline{\alpha} = -\underline{a}^{(l-n+1)}.$$

The poles of $[l, n]_f$ are then the roots of $P_2(z^{-1})$. Equivalently it can be easily shown (see e.g. [4]) that the poles are the generalized eigenvalues of $\left(U^{(q+1)}, U^{(q)}\right)$, $q = l - n + 1$, i.e. they are the roots of

$$det\left(U^{(q+1)} - zU^{(q)}\right) = 0. \tag{6}$$

In the case considered here the sequence $\{a_k\}$ is a sequence of i.i.d. random variables. The Padé approximants $[l, n]_f$ of the random function (3) exist a.s. as proved in [14].

Let us now recall the definition of the condensed density of the zeros of a random polynomial. Let us consider first the normalized counting measure

$$v_n = \frac{1}{n} \sum_{P_n(z)=0} \delta(z)$$

on the zeros λ_j of a deterministic polynomial $P_n(z)$. We can also define a spectral density

$$\rho_n(z) = \frac{1}{n} \sum_{j=1}^{n} \delta(z - \lambda_j).$$

If A is a Borel subset of $\mathbb{C}$ and $H = \{\lambda_1, \ldots, \lambda_n\}$ then

$$v_n(A) = \int_A \rho_n(z)\, dz.$$

When we are dealing with random polynomials the zeros counting measure v_n becomes a random measure. In this case, we are interested in the mean percentage of zeros belonging to A. Following [11] we can then define a condensed density $h_n(z)$ by

$$E\left[v_n(A)\right] = \int_A h_n(z)\, dz$$

which exists whatever is the joint distribution of the polynomial coefficients. Therefore,

$$h_n(z) = E[\rho_n(z)] = \frac{1}{n} E\left[\sum_{j=1}^{n} \delta(z - \lambda_j)\right].$$

We also have that, when the random polynomial leading coefficient is equal to one, all the marginal densities of the roots $\lambda_1, \ldots, \lambda_n$ are equal to the condensed density function (see e.g. [5]). We can now prove the basic result where we denote by tr the trace operator and by $\log_M$ the matrix logarithm operator.

Theorem 1. *The condensed density of the zeros of the random polynomial*

$$P_2(z^{-1}) = Q(z) = \det\left(U^{(q+1)} - zU^{(q)}\right)$$

is given by

$$h_n(z) = \frac{1}{4\pi} \Delta u_n(z), \tag{7}$$

where $z = x + iy$, $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, *and*

$$u_n(z) = \frac{1}{n} E \left\{ \mathrm{tr} \left(\log_M \left[(U^{(q+1)} - zU^{(q)}) \overline{(U^{(q+1)} - zU^{(q)})} \right] \right) \right\}$$

or, equivalently,

$$u_n(z) = \frac{1}{n} E \left\{ \log(|Q(z)|^2) \right\}. \tag{8}$$

Proof. From the definition of matrix logarithm we have

$$\log[\det(A)] = \mathrm{tr}[\log_M(A)]$$

for any square matrix A. Hence

$$\begin{aligned} \log|Q(z)|^2 &= \log[\det(U^{(q+1)} - zU^{(q)}) \cdot \overline{\det(U^{(q+1)} - zU^{(q)})}] \\ &= \log[\det\{(U^{(q+1)} - zU^{(q)}) \cdot (\overline{U^{(q+1)} - zU^{(q)}})\}] \\ &= \mathrm{tr} \log_M[(U^{(q+1)} - zU^{(q)}) \cdot (\overline{U^{(q+1)} - zU^{(q)}})]. \end{aligned}$$

But we also have

$$Q(z) = \det(U^{(q+1)} - zU^{(q)}) = \prod_j (\lambda_j - z) \cdot \det(U^{(q)}).$$

Hence

$$\begin{aligned} \log|Q(z)|^2 &= \log \prod_j |z - \lambda_j|^2 + \log|\det(U^{(q)})|^2 \\ &= \sum_j \log|z - \lambda_j|^2 + \mathrm{tr}\ \log_M(U^{(q)}\overline{U^{(q)}}). \end{aligned}$$

By equating the two expressions for $\log|Q(z)|^2$ we get

$$\begin{aligned} &\sum_j \log|z - \lambda_j|^2 \\ &\quad = \mathrm{tr}\ \log_M[(U^{(q+1)} - zU^{(q)}) \cdot (\overline{U^{(q+1)} - zU^{(q)}})] - \mathrm{tr} \log_M(U^{(q)}\overline{U^{(q)}}). \end{aligned} \tag{9}$$

Let us now consider the classic result [19, p. 47]:

$$\Delta\ \log \frac{1}{\sqrt{(x^2 + y^2)}} = -2\pi\delta(x)\delta(y). \tag{10}$$

But then if $z = x + iy$

$$\frac{1}{4\pi} \Delta\ \log(|z|^2) = \delta(z).$$

Using this identity we have:

$$h_n(z) = \frac{1}{n} \sum_{j=1}^{n} E\left[\delta(z - \lambda_j)\right] = \frac{1}{4\pi n} \sum_{j=1}^{n} E\left[\Delta \log|z - \lambda_j|^2\right].$$

But $\Delta \log|z-\lambda_j|^2$ is a nonnegative distribution, therefore if $\phi(z)$ is a nonnegative test function with compact support Ω, then by definition

$$\Delta \log|z-\lambda_j|^2(\phi) = \int_\Omega (\Delta\phi(z)) \log|z-\lambda_j|^2\, dz$$

and

$$\int_\Omega h_n(z)\phi(z)\, dz = \frac{1}{4\pi n}\sum_{j=1}^n E\left[\int_\Omega (\Delta\phi(z)) \log|z-\lambda_j|^2\, dz\right] < \infty$$

because $\phi(z)$ is continuous and $\int_{\mathbb{C}} h_n(z)\, dz = 1$, hence by Tonelli's theorem

$$\int_\Omega h_n(z)\phi(z)\, dz = \frac{1}{4\pi n}\sum_{j=1}^n \int_\Omega (\Delta\phi(z)) E\left[\log|z-\lambda_j|^2\right] dz.$$

Therefore $h_n(z) = \frac{1}{4\pi n}\sum_{j=1}^n \Delta E[\log|z-\lambda_j|^2]$. But then, from (9), we have

$$h_n(z) = \frac{1}{4\pi n}\Delta E\left\{\mathrm{tr}\ \log_M\left[(U^{(q+1)} - zU^{(q)})\overline{(U^{(q+1)} - zU^{(q)})}\right]\right\}$$

because $E\left\{\mathrm{tr}\left(\log_M\left[U^{(q)}\overline{U^{(q)}}\right]\right)\right\}$ does not depend on z. This completes the proof. $\square$

Remark. We notice that $-\frac{1}{4\pi}u_n(z)$ is the logarithmic potential of the condensed density $h_n(z)$ [13].

In the next theorem the circular symmetry of the condensed density $h_n(z)$ and its independence of the noise variance σ^2 are proved.

In the following, in order to simplify the notations, the functions are denoted by the same symbols either when a change of variables from cartesian to polar coordinates is done or when the dependence on a specific variable has to be stressed.

Theorem 2. *If the data a_k, $k = q, \ldots, q+2n-1$ are independent, zero mean complex Gaussian random variables with variance σ^2 and $z = re^{i\alpha}$ then $u_n(z)$ is a function of r alone and is given by*

$$u_n(r) = \frac{1}{n} E\left\{\log\left(\det[F(r, \underline{a})]\right)\right\}, \tag{11}$$

where

$$F(r, \underline{a}) = U^{(q+1)}\overline{U^{(q+1)}} + r^2 U^{(q)}\overline{U^{(q)}} - r[U^{(q)}\overline{U^{(q+1)}} + U^{(q+1)}\overline{U^{(q)}}] \tag{12}$$

and

$$\underline{a} = [a_q, \ldots, a_{q+2n-1}].$$

Moreover the condensed density in polar coordinates is

$$h_n(r,\alpha) = \frac{r}{4\pi}\left(\frac{\partial^2 u_n}{\partial r^2} + \frac{1}{r}\frac{\partial u_n}{\partial r}\right), \tag{13}$$

the marginal condensed density with respect to r is

$$h_n^{(r)}(r) = \frac{1}{2}\left(r u_n'' + u_n'\right) \tag{14}$$

and the marginal condensed density with respect to α *is uniform on* $[-\pi, \pi]$. *Moreover* $h_n^{(r)}(r)$ *is independent of* σ^2.

Proof. By hypothesis

$$\underline{a} \sim \frac{1}{(\pi\sigma^2)^N}\, e^{-\frac{|\underline{a}|^2}{\sigma^2}}, \text{ where } N = 2n.$$

This density is then invariant under the transformation

$$\underline{a} \to e^{\pm i\frac{\beta}{2}}\underline{a}. \tag{15}$$

Let us show that if $\tilde{z} = e^{i\beta}z = re^{i(\alpha+\beta)}$ then $u_n(z) = u_n(\tilde{z})\ \forall\beta$. From (8) we have

$$\begin{aligned}
u_n(\tilde{z}) &= \frac{1}{n}\, E\left\{\log[\det\{(U^{(q+1)} - re^{i(\alpha+\beta)}U^{(q)}) \cdot (\overline{U^{(q+1)} - re^{i(\alpha+\beta)}U^{(q)}})\}]\right\}\\
&= \frac{1}{n}\, E\left\{\log[\det\{U^{(q+1)}\overline{U^{(q+1)}} + r^2 U^{(q)}\overline{U^{(q)}}\right.\\
&\qquad \left. -r(e^{i(\alpha+\beta)}U^{(q)}\overline{U^{(q+1)}} + e^{-i(\alpha+\beta)}U^{(q+1)}\overline{U^{(q)}})\}]\right\}.
\end{aligned}$$

Letting $V^{(q)} = e^{i\beta/2}U^{(q)}$, $V^{(q+1)} = e^{-i\beta/2}U^{(q+1)}$ we have

$$\begin{aligned}
&V^{(q)}\overline{V^{(q)}} = U^{(q)}\overline{U^{(q)}}, \quad V^{(q+1)}\overline{V^{(q+1)}} = U^{(q+1)}\overline{U^{(q+1)}},\\
&e^{i(\alpha+\beta)}U^{(q)}\overline{U^{(q+1)}} = e^{i\alpha}(e^{i\beta/2}U^{(q)})(\overline{e^{-i\beta/2}U^{(q+1)}}) = e^{i\alpha}V^{(q)}\overline{V^{(q+1)}},\\
&e^{-i(\alpha+\beta)}U^{(q+1)}\overline{U^{(q)}} = e^{-i\alpha}(e^{-i\beta/2}U^{(q+1)})(\overline{e^{i\beta/2}U^{(q)}}) = e^{-i\alpha}V^{(q+1)}\overline{V^{(q)}}
\end{aligned}$$

and

$$\begin{aligned}
u_n(\tilde{z}) = \frac{1}{n}\, E&\left\{\log[\det\{V^{(q+1)}\overline{V^{(q+1)}} + r^2 V^{(q)}\overline{V^{(q)}}\right.\\
&\left. -r(e^{i\alpha}V^{(q)}\overline{V^{(q+1)}} + e^{-i\alpha}V^{(q+1)}\overline{V^{(q)}})\}]\right\}.
\end{aligned}$$

Hence, from the invariance property of the distribution of $\underline{a}$ under transformation (15), we have $u_n(\tilde{z}) = u_n(z)$. But then taking $\beta = -\alpha$ we have $u_n(\tilde{z}) = u_n(r)$.

In order to prove (13) and (14) let us consider polar coordinates $x = r\cos\alpha$, $y = r\sin\alpha$ in (7). Remembering that the Laplacian $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, in polar coordinates is

$$\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \alpha^2} + \frac{1}{r}\frac{\partial}{\partial r}$$

we get

$$h_n(r,\alpha) = r h_n(r\cos\alpha, r\sin\alpha) = \frac{r}{4\pi}\left(\frac{\partial^2 u_n}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u_n}{\partial \alpha^2} + \frac{1}{r}\frac{\partial u_n}{\partial r}\right). \tag{16}$$

Therefore, as u_n does not depend on α, formula (16) reduces to

$$h_n(r,\alpha) = \frac{r}{4\pi}\left(\frac{\partial^2 u_n}{\partial r^2} + \frac{1}{r}\frac{\partial u_n}{\partial r}\right) = h_n(r). \tag{17}$$

The marginal densities with respect to r and α are, respectively,

$$h_n^{(r)}(r) = \int_{-\pi}^{\pi} h_n(r,\alpha)\,d\alpha = \int_{-\pi}^{\pi} h_n(r)\,d\alpha = 2\pi h_n(r) = \frac{1}{2}\left(r u_n'' + u_n'\right), \tag{18}$$

and

$$h_n^{(\alpha)}(\alpha) = \int_0^{\infty} h_n(r,\alpha)\,dr = \int_0^{\infty} h_n(r)\,dr = \frac{1}{2\pi}\int_0^{\infty} h_n^{(r)}(r)\,dr = \frac{1}{2\pi}$$

therefore $h_n^{(\alpha)}(\alpha)$ has a uniform density on $[-\pi,\pi]$.

We show now that $h_n^{(r)}(r)$ is independent of σ^2. Let us consider the change of variable $\underline{\tilde{a}} = \frac{1}{\sigma}\underline{a}$ and write $u_n(r)$ as

$$u_n(r) = \frac{1}{n(\pi\sigma^2)^N}\int \log\left(\det\{F(r,\underline{a})\}\right) e^{-\frac{|\underline{a}|^2}{\sigma^2}}\,d\underline{a}. \tag{19}$$

But

$$d\underline{a} = \prod_{k=0}^{N-1} \Re da_k \cdot \prod_{k=0}^{N-1} \Im da_k = \sigma^N \prod_{k=0}^{N-1} \Re d\tilde{a}_k \cdot \sigma^N \prod_{k=0}^{N-1} \Im d\tilde{a}_k = \sigma^{2N}\,d\underline{\tilde{a}},$$

where $\Re$ and $\Im$ denote real and imaginary parts. Moreover $F(r,\underline{a}) = \sigma^2 F(r,\underline{\tilde{a}})$, therefore $\det\{F(r,\underline{a})\} = \sigma^{2N}\det\{F(r,\underline{\tilde{a}})\}$ and

$$u_n(r) = 2N\,\log(\sigma) + \frac{1}{n\pi^N}\int \log\left(\det\{F(r,\underline{\tilde{a}})\}\right) e^{-|\underline{\tilde{a}}|^2}\,d\underline{\tilde{a}}. \tag{20}$$

But then u_n', u_n'' and hence $h_n^{(r)}(r)$ do not depend on σ. $\square$

In the next theorem we prove that the condensed density $h_n^{(r)}(r)$ converges to the uniform measure on the unit circle when $n \to \infty$.

Theorem 3. $\lim_{n\to\infty} h_n^{(r)}(r) = \delta(r-1)$.

Proof. Let us consider the matrix function $g(A) = \log(\det[A])$ defined on the Hermitian matrices A. From [17, F.2.c, p. 476] we know that g is a matrix-concave function. Hence from a generalization of Jensen inequality to matrix functions [17, E.6, p. 467] it follows that

$$E\left\{\log\left(\det[F(r,\underline{a})]\right)\right\} \leqslant \log\left(\det[E\left\{F(r,\underline{a})\right\}]\right).$$

But

$$\begin{aligned} E[F] = E[U^{(q+1)}\overline{U^{(q+1)}}] + r^2 E[U^{(q)}\overline{U^{(q)}}] \\ -r\left(E[U^{(q)}\overline{U^{(q+1)}}] + E[U^{(q+1)}\overline{U^{(q)}}]\right) \end{aligned}$$

and, because the a_k are independent random variables,

$$\underline{e}_j^T E[U^{(q+1)}\overline{U^{(q+1)}}]\underline{e}_k = \sum_{h=1}^{n} E[a_{j+h}\overline{a}_{h+k}] = \delta_{j,k} n\sigma^2,$$

$$\underline{e}_j^T E[U^{(q)}\overline{U^{(q)}}]\underline{e}_k = \sum_{h=1}^{n} E[a_{j+h-1}\overline{a}_{h+k-1}] = \delta_{j,k} n\sigma^2,$$

$$\underline{e}_j^T E[U^{(q+1)}\overline{U^{(q)}}]\underline{e}_k = \sum_{h=1}^{n} E[a_{j+h}\overline{a}_{h+k-1}] = \delta_{j,k-1} n\sigma^2,$$

$$\underline{e}_j^T E[U^{(q)}\overline{U^{(q+1)}}]\underline{e}_k = \sum_{h=1}^{n} E[a_{j+h-1}\overline{a}_{h+k}] = \delta_{j-1,k} n\sigma^2.$$

Hence

$$E[F] = n\sigma^2 \begin{bmatrix} 1+r^2 & -r & 0 & \dots & 0 \\ -r & 1+r^2 & -r & 0 & \dots \\ . & . & . & . & . \\ 0 & \dots & 0 & -r & 1+r^2 \end{bmatrix}.$$

Moreover it is easy to show that $\det(E[F]) = (n\sigma^2)^n \sum_{j=0}^{n} r^{2j}$. Hence

$$\log\det(E[\underline{f}]) = n\,\log(n\sigma^2) + \log\sum_{j=0}^{n} r^{2j}$$

and

$$u_n(r) \leqslant \log(n\sigma^2) + \frac{1}{n}\,\log\sum_{j=0}^{n} r^{2j}. \tag{21}$$

In order to find a lower bound for $u_n(r)$ let us rewrite the marginal condensed density with respect to r as $h_n^{(r)}(r) = [\frac{1}{2} r u_n']'$. Hence $\frac{1}{2} r u_n'$ must be a distribution function. But then $\frac{1}{2} r u_n' \geqslant 0$ and therefore $u_n(r)$ is nondecreasing. Moreover, it is easy to show that

$$u_n(0) = c_n > -\infty \tag{22}$$

and

$$\lim_{r\to\infty} \frac{u_n(r)}{\log(r^2)} = 1. \tag{23}$$

But then the Green function of the differential operator in (14) with the boundary conditions (22) and (23) is

$$G(r,x) = \begin{cases} \log x^2 & \text{if } r \leqslant x, \\ \log r^2 & \text{if } r > x, \end{cases}$$

therefore,

$$u_n(r) = \int_0^\infty G(r,x) h_n^{(r)}(x)\, dx = \log r^2 \int_0^r h_n^{(r)}(x)\, dx + \int_r^\infty \log x^2 h_n^{(r)}(x)\, dx.$$

Hence if $r \geqslant 1$

$$u_n(r) \geqslant \log r^2 \int_0^\infty h_n^{(r)}(x)\, dx = \log r^2$$

as $h_n^{(r)}(x)$ is a probability density. Moreover as $u_n(r)$ is not decreasing we have that for $0 \leqslant r < 1$, $u_n(r) \geqslant u_n(0) = c_n$. Hence a lower bound for $u_n(r)$ is

$$v_n(r) = \begin{cases} c_n & \text{if } 0 \leqslant r \leqslant 1, \\ \log r^2 & \text{if } r > 1. \end{cases}$$

After Theorem 2 we know that $h_n^{(r)}(r)$ does not depend on σ^2 therefore we can choose a variance dependent on n such that $\log(n\sigma_n^2) = c_n$. From (21) an upper bound for $u_n(r)$ is then given by

$$w_n(r) = c_n + \frac{1}{n} \log \sum_{j=0}^{n} r^{2j}.$$

It is easy to show that for $r > 1$

$$\lim_{n\to\infty} \frac{1}{n} \log \sum_{j=0}^{n} r^{2j} = \log r^2.$$

Hence the sequence c_n is bounded above by zero and $\lim_{n\to\infty} c_n = 0$ because of the continuity of u_n. Hence

$$\lim_{n\to\infty} w_n(r) = \lim_{n\to\infty} v_n(r). \tag{24}$$

Moreover, we notice that

$$w_n'(r) = \frac{1}{n}\frac{\sum_{j=1}^n 2jr^{2j-1}}{\sum_{j=0}^n r^{2j}} = \frac{2r}{1-r^2}\frac{nr^{2n}(1-r^2)+r^{2n}-1}{n(r^{2n}r^2-1)}$$

therefore

$$w_\infty'(r) = \lim_{n\to\infty} w_n'(r) = \begin{cases} 0 & \text{if } 0\leqslant r\leqslant 1 \\ \frac{2}{r} & \text{if } r>1 \end{cases} = \begin{cases} 0 & \text{if } 0\leqslant r\leqslant 1, \\ 2H(r-1)+\frac{2(1-r)}{r} & \text{if } r>1, \end{cases}$$

where $H(r)$ is the Heaviside function. Moreover

$$w_\infty''(r) = \lim_{n\to\infty} w_n''(r) = \begin{cases} 0 & \text{if } 0\leqslant r\leqslant 1, \\ 2\delta(r-1)-\frac{2}{r^2} & \text{if } r>1. \end{cases}$$

But, by considering $v_n(r)$ as a distribution, we have $\lim_{n\to\infty} v_n'(r) = w_\infty'(r)$ and $\lim_{n\to\infty} v_n''(r) = w_\infty''(r)$. Therefore, because of (24) we must have

$$\lim_{n\to\infty} u_n'(r) = w_\infty'(r), \qquad \lim_{n\to\infty} u_n''(r) = w_\infty''(r)$$

hence

$$\lim_{n\to\infty} h_n^{(r)}(r) = \lim_{n\to\infty} \frac{1}{2}\left(ru_n''+u_n'\right) = r\delta(r-1) = \delta(r-1). \qquad \square$$

2. The condensed density model

In order to get from (14) an analytic expression for the marginal condensed density with respect to r, we need an analytic expression for the function $u_n(r)$ which is unfortunately given by the difficult integral (19) which is not computable for general n by standard methods. We then look for an analytic model $\tilde{h}_n^{(r)}(r)$ of the condensed density $h_n^{(r)}(r)$ which is consistent with the empirical distributions that can be obtained by Monte Carlo simulations. To this aim we start by noticing that the upper bound $w_1(r)$ gives rise to the true condensed density $h_1^{(r)}(r)$. In fact we have

Theorem 4. $u_1(r) = \frac{1}{2}(\log\sigma^2 - \Gamma) + \log(1+r^2)$, *where* Γ *is the Euler constant. Moreover* $h_1^{(r)}(r) = \frac{2r}{(1+r^2)^2}$.

Proof. When $n = 1$ the generalized eigenvalue problem (6) becomes

$$a_{q+1} - za_q = 0.$$

From standard results of interpolation by sum of complex exponentials [12, Theorem 2.c], there exist random variables (λ, b) such that

$$b = a_q, \quad \lambda b = a_{q+1}.$$

But by hypothesis (a_{q+1}, a_q) has a bivariate complex Gaussian density

$$(a_{q+1}, a_q) \sim \frac{1}{(\pi\sigma^2)^2} e^{-\frac{|a_{q+1}|^2+|a_q|^2}{\sigma^2}}.$$

By making the transformation

$$T : (a_{q+1}, a_q) \to (\lambda, b)$$

noting that the complex Jacobian of T^{-1} is $J_C = b$ and remembering that the Jacobian with respect to the real and imaginary part of the complex variables is $J_R = |J_C|^2 = |b|^2$ (see e.g. the appendix in [5]), we get

$$(\lambda, b) \sim \frac{1}{(\pi\sigma^2)^2} e^{-\frac{|b|^2(1+|\lambda|^2)}{\sigma^2}} |b|^2.$$

The marginal density with respect to λ is then

$$\frac{1}{(\pi\sigma^2)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{|b|^2(1+|\lambda|^2)}{\sigma^2}} |b|^2 \cdot d\Re b \cdot d\Im b = \frac{1}{\pi(1+|\lambda|^2)^2}.$$

In polar coordinates $\lambda = re^{i\alpha}$ we get

$$h_1^{(r)}(r) = \int_{-\pi}^{\pi} \frac{1}{\pi(1+r^2)^2}\, r\, d\alpha = \frac{2r}{(1+r^2)^2}$$

hence $u_1(r)$ must satisfy the ODE

$$\frac{1}{2}\left(ru_1'' + u_1'\right) = \frac{2r}{(1+r^2)^2}$$

whose general solution is

$$c_1 + c_2 \log(r) + \log(1+r^2). \tag{25}$$

But when $n = 1$ and $r = 0$ the integral in (19) can be done explicitly giving

$$u_1(0) = \frac{1}{2}(\log\sigma^2 - \Gamma).$$

Taking the limit for $r \to 0$ in (25) and imposing this initial condition and the condition that the solution is bounded in zero, we get the result. $\square$

Motivated by this result, we propose a class of models which coincides with the true density for $n = 1$ and has the expected asymptotic behavior as prescribed by Theorem 3.

Let us first define

$$\hat{u}_n(r) = \frac{1}{n} \log \left(\sum_{j=0}^{n} \gamma_j^{(n)} r^{2j} \right),$$

where $\gamma_j^{(n)} = \gamma_{n-j}^{(n)}$, $j = 0, \ldots, n$; $0 < \gamma_j^{(n)} \leqslant 1$, and $\gamma_0^{(n)} = \gamma_n^{(n)} = 1$ in order to correctly reproduce $u_1(r)$ (apart from a constant). Then define

$$\hat{h}_n^{(r)}(r) = \frac{1}{2} \left(r\hat{u}_n'' + \hat{u}_n' \right).$$

The following theorem holds

Theorem 5. *The model* $\hat{h}_n^{(r)}(r)$ *is a probability density i.e. it is nonnegative and* $\int_0^\infty \hat{h}_n^{(r)}(r)\, dr = 1$. *Moreover* $\lim_{n\to\infty} \hat{h}_n^{(r)}(r) = \delta(r-1)$.

Proof. Let be $H_n(r) = \frac{1}{2} r\hat{u}_n'(r)$. As $H_n'(r) = \hat{h}_n^{(r)}(r)$ it is enough to show that $H_n(r)$ is a nonnegative nondecreasing function such that $H_n(0) = 0$ and $\lim_{r\to\infty} H_n(r) = 1$. From the definition of $\hat{u}_n(r)$ we have

$$H_n(r) = \frac{\frac{1}{n}\sum_{j=1}^{n-1} j\gamma_j^{(n)} r^{2j} + r^{2n}}{1 + \sum_{j=1}^{n-1} \gamma_j^{(n)} r^{2j} + r^{2n}}$$

hence $H_n(r)$ is nonnegative, $H_n(0) = 0$ and $\lim_{r\to\infty} H_n(r) = 1$. Let us show that it is nondecreasing. Let be $r < \tilde{r}$; it is easy to show that $H_n(r) - H_n(\tilde{r}) = \sum_k \alpha_k (r^{2k} - \tilde{r}^{2k})$ where $\alpha_k > 0$. Hence $H_n(r)$ is nondecreasing. The last part of the thesis follows by the same arguments used in the proof of Theorem 3, noticing that $\hat{u}_n(r) \leqslant w_n(r)$ because $\gamma_j^{(n)} \leqslant 1$. □

Remark. We notice that, as $h_n^{(r)}(r)$, also $\hat{h}_n^{(r)}(r)$ does not depend on the noise variance σ^2.

3. Numerical results

In this section we report on some Monte Carlo simulations for computing the empirical distributions of the absolute value of the poles of the Padé approximants $[l, n]_f$ for several orders and compare them to a specific model from the class described in the previous section.

By choosing

$$\gamma_j^{(n)} = (1 + \log j)^{-(1+\beta_n)}, \quad \beta_n = \log(1 + \log(1 + 0.01 * n)),$$
$$j = 1, \ldots, n-1, \quad n = 2, 3, \ldots$$

we got a good fit with the empirical distributions in the range $n \in (1, 500)$ as shown in Figs. 2–4 where the empirical distributions of the absolute value of the poles of the Padé

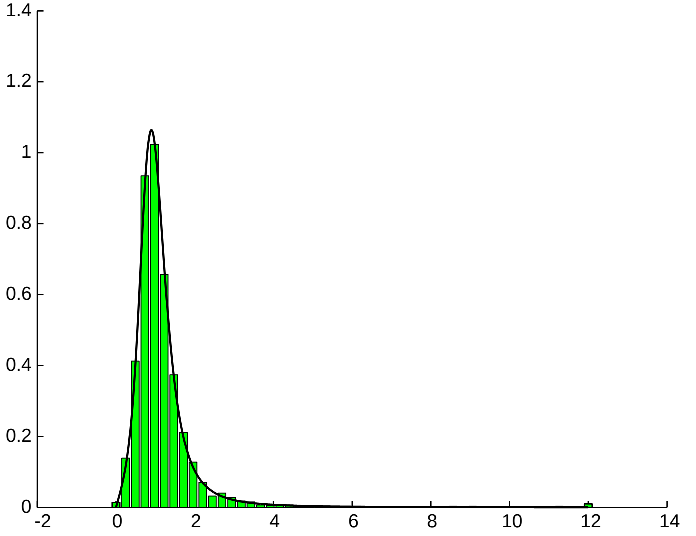

Fig. 2. Empirical distribution of the modulus of the poles of $[2, 3]_v(z)$; solid: $\hat{h}_3^{(r)}(r)$.

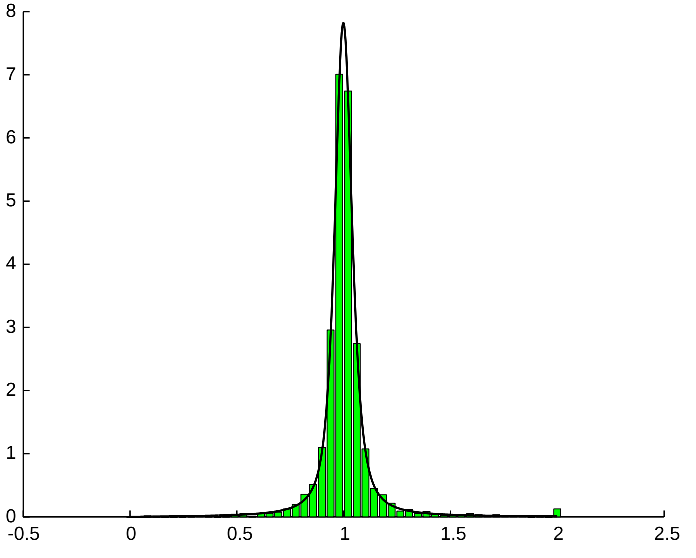

Fig. 3. Empirical distribution of the modulus of the poles of $[29, 30]_v(z)$; solid: $\hat{h}_{30}^{(r)}(r)$.

approximants of order [2, 3], [29, 30], [499, 500], respectively, are shown. The empirical distributions are built using 3000 samples each. In Fig. 5 the model function $\hat{h}_n^{(r)}(r)$ is plotted for several values of n.

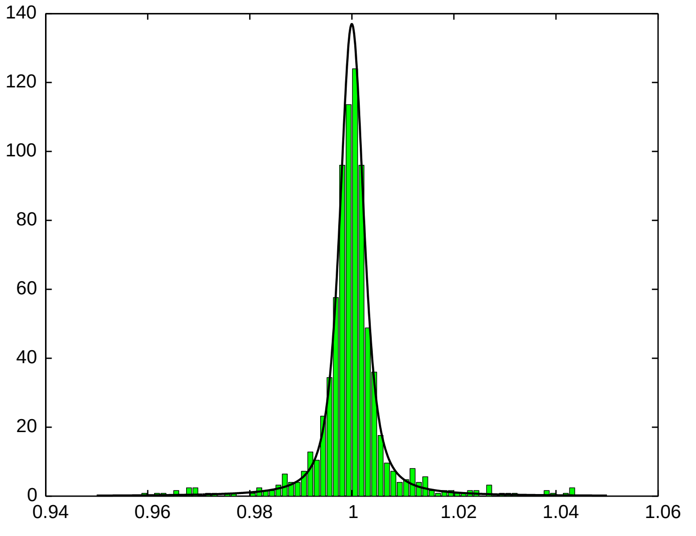

Fig. 4. Empirical distribution of the modulus of the poles of $[499, 500]_v(z)$; solid: $\hat{h}_{500}^{(r)}(r)$.

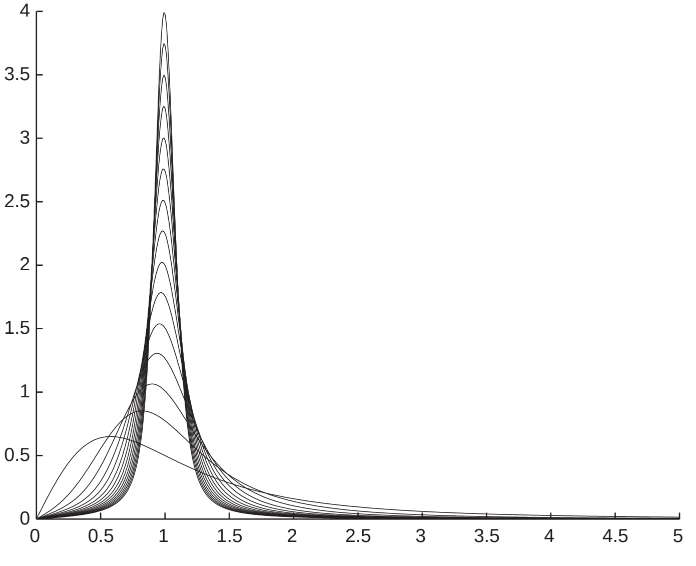

Fig. 5. The functions $\hat{h}_n^{(r)}(r)$ for $n = 1, 2, \ldots, 15$.

Acknowledgments

The author is grateful to G.L. Torrisi for his valuable comments and careful reading of the manuscript.

References

[1] G.A. Baker Jr., Essentials of Padé Approximants, Academic Press, New York, 1975.
[2] P. Barone, R. March, On the super-resolution properties of Prony's method, ZAMM Z. Angew. Math. Mech. 76 (1996) 177–180.
[3] P. Barone, R. March, Some properties of the asymptotic location of poles of Padé approximants to noisy rational functions, relevant for modal analysis, IEEE Trans. Signal Process. 46 (1998) 2448–2457.
[4] P. Barone, R. March, A novel class of Padé based method in spectral analysis, J. Comput. Methods Sci. Eng. 1 (2001) 185–211.
[5] P. Barone, Orthogonal polynomials, random matrices and the numerical inversion of Laplace transform of positive functions, J. Comput. Appl. Math. 155 (2003) 307–330.
[6] D. Bessis, Padé approximants in noise filtering, J. Comput. Appl. Math. 66 (1996) 85–88.
[7] T. Carleman, Les Functions Quasi Analytiques, Gauthier–Villars, Paris, 1926.
[8] J.L. Gammel, Effect of random errors (noise) in the terms of a power series on the convergence of the Padé approximants, in: P.R. Graves-Morris (Ed.), Padé Approximants, The Institute of Physics, London and Bristol, 1972.
[9] J.L. Gammel, J. Nuttall, Convergence of Padé approximants to quasianalytic functions beyond natural boundaries, J. Math. Anal. Appl. 43 (1973) 694–696.
[10] J. Gilewicz, Approximants de Padé, Lecture Notes in Mathematics, vol. 667, Springer, Berlin, 1978.
[11] J.M. Hammersley, The zeros of a random polynomial, Proceedings of the Berkeley Symposium on Mathematical and Statistical Probability, vol. 2, 1956, pp. 89–111.
[12] P. Henrici, Applied and Computational Complex Analysis, vol. II, Wiley, New York, 1977.
[13] O.D. Kellogg, Fondations of Potential Theory, Springer, Berlin, 1929.
[14] Li Jialiang, On the existence and convergence of random Padé approximants, Adv. Math. 22 (1993) 340–347.
[15] R. March, P. Barone, Application of the Padé method to solve the noisy trigonometric moment problem: some initial results, SIAM J. Appl. Math. 58 (1998) 324–343.
[16] R. March, P. Barone, Reconstruction of a piecewise constant function from noisy Fourier coefficients by Padé method, SIAM J. Appl. Math. 60 (2000) 1137–1156.
[17] A.W. Marshall, I. Olkin, Inequalities: Theory of Majorization and its Applications, Academic Press, New York, 1979.
[18] L. Pakula, Asymptotic zero distribution of orthogonal polynomials in sinusoidal frequency estimation, IEEE Trans. Inform. Theory 33 (1987) 569–576.
[19] L. Schwartz, Théorie des distributions, vol. 1, Hermann, Paris, 1950.